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Entangled state representations in non-commutative space and their applications

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Abstract
We introduce new representations to formulate quantum mechanics on non-commutative coordinate space, which explicitly display entanglement properties between degrees of freedom of different coordinate components and hence could be called entangled state representations. Furthermore, we derive unitary transformations between the new representations and the ordinary ones used in non-commutative quantum mechanics (NCQM). To show the potential applications of the entangled state representations, a two-dimensional harmonic oscillator on the non-commutative plane with both coordinate–coordinate and momentum–momentum couplings is exactly solved.

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1. Introduction

As is well known, representations and transformation theories, founded by Dirac [1], play a basic and important role in quantum mechanics. Many quantum mechanics problems were solved cleverly by working in specific representations. Some representations, such as, the coordinate, the momentum, the number representation, as well as the coherent state representation, are often employed in the literature of ordinary quantum mechanics. In non-commutative quantum mechanics (NCQM) [2], because of the non-commutativity of coordinate-component operators, there are no common eigenstates for these different coordinate operators, and one can hardly construct a coordinate representation in the usual sense. However, in order to formulate quantum mechanics on a non-commutative space so that some dynamic problems can be solved, we do need some appropriate representations. On the other hand, we realize that in NCQM ordinary products are usually replaced by $*$-products between functions on the non-commutative space [3], which is equivalent to working in a 'commuting coordinate representation' (in this representation, the state vectors, for example, $|x, y\rangle$, are not spontaneously eigenstates of the coordinate operators $\hat{X}$ and $\hat{Y}$ in the non-commutative space; for details, see section 4). Of course, if we have more practical representations for NCQM, it will be more effective to deal with the dynamic problems in NCQM.

Noting that although two coordinate-component operators on the non-commutative space do not commute with each other, the difference of the two coordinate operators indeed commute with the sum of the relevant two momentum operators, thus we can still employ Einstein–Podolsky–Rosen's (EPR) [4] idea to construct entangled states on the non-commutative space. The problem of entanglement itself is very important in both fundamental and applied investigation and now is intensively studied [5]. In most of these studies, the variables describing entanglements are discrete. For the case of continuous variables there exist many open questions[3]. It is easy to show that the entangled states with continuum variables are orthonormal and satisfy completeness relations, therefore they present new representations for NCQM. The first bipartite entangled state representation of continuum variables is constructed by one of the authors (H Fan) and J R Klauder [6]. Here we extend the formalism in [6] to NCQM and investigate some basic properties of the entangled state representations on the non-commutative space. We also derive explicit unitary operators which connect the entangled state representations and the 'commuting coordinate representation' and transform them into each other. To show the potential applications of the entangled state representations in NCQM, we solve exactly the energy level of a two-dimensional harmonic oscillator on a non-commutative plane with both kinetic coupling and elastic coupling.

The work is arranged as follows. In section 2 we construct the entangled state representations for NCQM and derive matrix elements of coordinate and momentum operators in these representations. In order to demonstrate that these states are indeed the entangled states, we study their Schmidt decompositions in section 3. In section 4 we investigate the transformation between the 'commuting coordinate' and the entangled state representations, and derive an explicit unitary operator which transforms them into each other. In section 5 we study a two-dimensional harmonic oscillator on a non-commutative plane with both kinetic coupling and elastic coupling and solve its energy spectrum exactly. The summary and discussion are presented in section 6.

2. Entangled state representations for NCQM

Without loss of generality and for the sake of simplicity, we only discuss the non-commutative plane case in what follows. Operators $\hat{X}$, $\hat{Y}$, $\hat{P}_x$ and $\hat{P}_y$ satisfy the following commutation relations,

$$[\hat{X}, \hat{Y}] = \mathrm{i}\theta, \qquad [\hat{X}, \hat{P}_x] = \mathrm{i}, \qquad [\hat{Y}, \hat{P}_y] = \mathrm{i}, \tag{1}$$

and other commutators of these operators are vanishing, where θ is a real parameter reflecting the non-commutativity of space coordinates, and we take $\hbar = 1$. Consider the following operators:

$$\hat{R} = \frac{\hat{X} - \hat{Y}}{\sqrt{2}}, \qquad \hat{P} = \frac{\hat{P}_x + \hat{P}_y}{\sqrt{2}}, \qquad \hat{S} = \frac{\hat{X} + \hat{Y}}{\sqrt{2}}, \qquad \hat{K} = \frac{\hat{P}_x - \hat{P}_y}{\sqrt{2}}. \tag{2}$$

Obviously $\hat{R}$ and $\hat{P}$ commute with each other, as well as $\hat{S}$ and $\hat{K}$ commute with each other. Thus $\hat{R}$ and $\hat{P}$ have common eigenstates $|\eta\rangle$, and $\hat{S}$ and $\hat{K}$ have common eigenstates $|\xi\rangle$. Here η and ξ may be complex numbers ($\eta = \eta_1 + \mathrm{i}\eta_2$ and $\xi = \xi_1 + \mathrm{i}\xi_2$) and η_1, η_2, ξ_1 and ξ_2 are real numbers.

[3] We thank the referee to draw our attention to this fact.

In order to get explicit expressions of the eigenstates $|\eta\rangle$ and $|\xi\rangle$, we use the following transformations,

$$\hat{X} = x - \frac{\theta}{2} p_y, \qquad \hat{Y} = y + \frac{\theta}{2} p_x, \qquad \hat{P}_x = p_x, \qquad \hat{P}_y = p_y, \tag{3}$$

where the operators x, y, p_x and p_y satisfy ordinary Heisenberg commutation relations,

$$[x, p_x] = \mathrm{i}, \qquad [y, p_y] = \mathrm{i}, \tag{4}$$

and other commutators of these operators are vanishing. Furthermore, introducing two independent ordinary bosonic creation and annihilation operators $a^\dagger, a$ and $b^\dagger, b$ with commutation relations $[a, a^\dagger] = 1$, $[b, b^\dagger] = 1$, we have

$$x = \frac{a + a^\dagger}{\sqrt{2}}, \qquad p_x = \frac{a - a^\dagger}{\sqrt{2}\mathrm{i}}, \qquad y = \frac{b + b^\dagger}{\sqrt{2}}, \qquad p_y = \frac{b - b^\dagger}{\sqrt{2}\mathrm{i}}. \tag{5}$$

In terms of these creation and annihilation operators, we can express the operators $\hat{X}$, $\hat{Y}$, $\hat{P}_x$ and $\hat{P}_y$ as

$$\begin{aligned} \hat{X} &= \frac{a + a^\dagger}{\sqrt{2}} - \frac{\theta(b - b^\dagger)}{2\sqrt{2}\mathrm{i}}, \qquad & \hat{P}_x &= \frac{a - a^\dagger}{\sqrt{2}\mathrm{i}}, \\ \hat{Y} &= \frac{b + b^\dagger}{\sqrt{2}} + \frac{\theta(a - a^\dagger)}{2\sqrt{2}\mathrm{i}}, \qquad & \hat{P}_y &= \frac{b - b^\dagger}{\sqrt{2}\mathrm{i}}. \end{aligned} \tag{6}$$

Thus the operators $\hat{R}$ and $\hat{P}$ may be expressed as

$$\hat{R} = \frac{1}{2}(a + a^\dagger - b - b^\dagger) - \frac{\theta}{4\mathrm{i}}(a - a^\dagger + b - b^\dagger), \qquad \hat{P} = \frac{1}{2\mathrm{i}}(a - a^\dagger + b - b^\dagger). \tag{7}$$

The common eigenstate $|\eta\rangle$ of $\hat{R}$ and $\hat{P}$ can be written as

$$|\eta\rangle = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{|\eta|^2}{2} + \eta a^\dagger - \eta^* b^\dagger + a^\dagger b^\dagger\right)|00\rangle, \tag{8}$$

where $|00\rangle$ is a two-mode bosonic vacuum state satisfying $a|00\rangle = 0$ and $b|00\rangle = 0$. It is easy to see that

$$\frac{1}{2}(a + a^\dagger - b - b^\dagger)|\eta\rangle = \eta_1|\eta\rangle, \qquad \frac{1}{2\mathrm{i}}(a - a^\dagger + b - b^\dagger)|\eta\rangle = \eta_2|\eta\rangle, \tag{9}$$

which leads to

$$\hat{R}|\eta\rangle = \left(\eta_1 - \frac{\theta}{2}\eta_2\right)|\eta\rangle, \qquad \hat{P}|\eta\rangle = \eta_2|\eta\rangle. \tag{10}$$

Here we would like to give an explicit proof of the completeness relation for the eigenstates $|\eta\rangle$ using a method of integration within ordered product (IWOP) [7]

$$\begin{aligned} \int_{-\infty}^{\infty} \mathrm{d}^2\eta |\eta\rangle\langle\eta| &= \int_{-\infty}^{\infty} \frac{\mathrm{d}^2\eta}{\pi} :\exp(-|\eta|^2 + \eta a^\dagger - \eta^* b^\dagger + a^\dagger b^\dagger - a^\dagger a - b^\dagger b + \eta^* a - \eta b + ab): \\ &=: \exp((a^\dagger - b)(a - b^\dagger) + a^\dagger b^\dagger + ab - a^\dagger a - b^\dagger b) := 1, \end{aligned} \tag{11}$$

where $\mathrm{d}^2\eta \equiv \mathrm{d}\eta_1\, \mathrm{d}\eta_2$ and we have used an expression $|00\rangle\langle 00| =: \exp(-a^\dagger a - b^\dagger b)$: and the notation $: \cdots :$ stands for taking the normal product of the creation and annihilation operators. It is easy to derive the inner product of the states $|\eta\rangle$

$$\langle\eta|\eta'\rangle = \delta^{(2)}(\eta - \eta') = \delta(\eta_1 - \eta_1')\delta(\eta_2 - \eta_2'). \tag{12}$$

Therefore, the eigenstates $|\eta\rangle$ form an orthonormal and complete set of base vectors and can be used to expand any other state vector in the related Hilbert space, so these states form a representation for NCQM.

Similarly, we may express the operators $\hat{S}$ and $\hat{K}$ as

$$\hat{S} = \frac{1}{2}(a + a^{\dagger} + b + b^{\dagger}) + \frac{\theta}{4\mathrm{i}}(a - a^{\dagger} - b + b^{\dagger}), \qquad \hat{K} = \frac{1}{2\mathrm{i}}(a - a^{\dagger} - b + b^{\dagger}). \tag{13}$$

The common eigenstate of $\hat{S}$ and $\hat{K}$ is

$$|\xi\rangle = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{|\xi|^2}{2} + \xi a^{\dagger} + \xi^* b^{\dagger} - a^{\dagger} b^{\dagger}\right)|00\rangle. \tag{14}$$

With the aid of two expressions

$$\frac{1}{2}(a + a^{\dagger} + b + b^{\dagger})|\xi\rangle = \xi_1|\xi\rangle, \qquad \frac{1}{2\mathrm{i}}(a - a^{\dagger} - b + b^{\dagger})|\xi\rangle = \xi_2|\xi\rangle, \tag{15}$$

we have

$$\hat{S}|\xi\rangle = \left(\xi_1 + \frac{\theta}{2}\xi_2\right)|\xi\rangle, \qquad \hat{K}|\xi\rangle = \xi_2|\xi\rangle. \tag{16}$$

Also the states $|\xi\rangle$ form an orthonormal and complete set of base vectors

$$\int_{-\infty}^{\infty} \mathrm{d}^2\xi\,|\xi\rangle\langle\xi| = 1, \qquad \langle\xi|\xi'\rangle = \delta^{(2)}(\xi - \xi') = \delta(\xi_1 - \xi_1')\delta(\xi_2 - \xi_2'), \tag{17}$$

where $\mathrm{d}^2\xi \equiv \mathrm{d}\xi_1\,\mathrm{d}\xi_2$.

Thus the eigenstates $|\eta\rangle$ and $|\xi\rangle$ form two representations for quantum mechanics on the non-commutative plane, respectively. In the next section, we will explain that in fact the states $|\eta\rangle$ and $|\xi\rangle$ basically are entangled states in the non-commutative plane, so we may call the $|\eta\rangle$ and $|\xi\rangle$ representations as entangled state representations. For the non-commutative quantum plane, sometimes working in the $|\eta\rangle$ or $|\xi\rangle$ representation is more convenient, so we first need to know the scalar product of $|\eta\rangle$ and $|\xi\rangle$. With the aid of overcompleteness of coherent states

$$\int \frac{\mathrm{d}^2 z_1\,\mathrm{d}^2 z_2}{\pi^2}|z_1, z_2\rangle\langle z_1, z_2| = 1, \tag{18}$$

where $|z_1, z_2\rangle$ is a two-mode canonical coherent state

$$|z_1, z_2\rangle = |z_1\rangle_a|z_2\rangle_b = \exp\left(-\tfrac{1}{2}(|z_1|^2 + |z_2|^2)\right)\exp(z_1 a^{\dagger} + z_2 b^{\dagger})|00\rangle, \tag{19}$$

one may simply get

$$\langle\eta|\xi\rangle = \int \frac{\mathrm{d}^2 z_1\,\mathrm{d}^2 z_2}{\pi^2}\langle\eta|z_1, z_2\rangle\langle z_1, z_2|\xi\rangle = \frac{1}{2\pi}\,\mathrm{e}^{\mathrm{i}(\eta_1\xi_2 - \eta_2\xi_1)}. \tag{20}$$

Having equation (20), one easily obtains all matrix elements of the basic operators $\hat{X}$, $\hat{Y}$, $\hat{P}_x$ and $\hat{P}_y$ on the non-commutative plane in the entangled state representation $|\eta\rangle$. To do this, we only need to evaluate $\langle\eta|\hat{S}|\eta'\rangle$ and $\langle\eta|\hat{K}|\eta'\rangle$, and obtain

$$\langle\eta|\hat{S}|\eta'\rangle = \langle\eta|\hat{S}\int \mathrm{d}^2\xi\,|\xi\rangle\langle\xi|\eta'\rangle = \mathrm{i}\left(\frac{\partial}{\partial\eta_2} - \frac{\theta}{2}\frac{\partial}{\partial\eta_1}\right)\delta^{(2)}(\eta - \eta'), \tag{21}$$

and

$$\langle\eta|\hat{K}|\eta'\rangle = \langle\eta|\hat{K}\int \mathrm{d}^2\xi\,|\xi\rangle\langle\xi|\eta'\rangle = -\mathrm{i}\frac{\partial}{\partial\eta_1}\delta^{(2)}(\eta - \eta'). \tag{22}$$

Thus in the $|\eta\rangle$ representation, we have

$$\begin{aligned}
\langle\eta|\hat{X}|\eta'\rangle &= \frac{1}{\sqrt{2}}\left(\eta_1 + \mathrm{i}\partial_{\eta_2} - \frac{\theta}{2}(\eta_2 + \mathrm{i}\partial_{\eta_1})\right)\delta^{(2)}(\eta - \eta'),\\
\langle\eta|\hat{Y}|\eta'\rangle &= \frac{-1}{\sqrt{2}}\left(\eta_1 - \mathrm{i}\partial_{\eta_2} - \frac{\theta}{2}(\eta_2 - \mathrm{i}\partial_{\eta_1})\right)\delta^{(2)}(\eta - \eta'),\\
\langle\eta|\hat{P}_x|\eta'\rangle &= \frac{1}{\sqrt{2}}\left(\eta_2 - \mathrm{i}\partial_{\eta_1}\right)\delta^{(2)}(\eta - \eta'),\\
\langle\eta|\hat{P}_y|\eta'\rangle &= \frac{1}{\sqrt{2}}\left(\eta_2 + \mathrm{i}\partial_{\eta_1}\right)\delta^{(2)}(\eta - \eta').
\end{aligned} \tag{23}$$

Similarly, in the $|\xi\rangle$ representation, we have

$$\begin{aligned}
\langle\xi|\hat{X}|\xi'\rangle &= \frac{1}{\sqrt{2}}\left(\xi_1 + \mathrm{i}\partial_{\xi_2} + \frac{\theta}{2}(\xi_2 + \mathrm{i}\partial_{\xi_1})\right)\delta^{(2)}(\xi - \xi'), \\
\langle\xi|\hat{Y}|\xi'\rangle &= \frac{1}{\sqrt{2}}\left(\xi_1 - \mathrm{i}\partial_{\xi_2} + \frac{\theta}{2}(\xi_2 - \mathrm{i}\partial_{\xi_1})\right)\delta^{(2)}(\xi - \xi'), \\
\langle\xi|\hat{P}_x|\xi'\rangle &= \frac{1}{\sqrt{2}}(\xi_2 - \mathrm{i}\partial_{\xi_1})\,\delta^{(2)}(\xi - \xi'), \\
\langle\xi|\hat{P}_y|\xi'\rangle &= \frac{-1}{\sqrt{2}}(\xi_2 + \mathrm{i}\partial_{\xi_1})\,\delta^{(2)}(\xi - \xi').
\end{aligned} \tag{24}$$

3. Entanglement properties of the states $|\eta\rangle$ and $|\xi\rangle$

From equations (8) and (14) we find that there exists an intrinsic entanglement of different degrees of freedom corresponding to different coordinate components on a non-commutative plane. Usually, these states are so-called entangled states, therefore we may name these two representations as entangled state representations. In order to show this kind of entanglement more explicitly, let us consider Fourier transform of the state $|\eta\rangle$. Using a familiar expression for eigenstate $|q\rangle$ of coordinate operator x ($x|q\rangle = q|q\rangle$) in Fock space

$$|q\rangle_a = \frac{1}{\sqrt[4]{\pi}}\exp\left(-\frac{q^2}{2} + \sqrt{2}qa^\dagger - \frac{a^{\dagger 2}}{2}\right)|0\rangle, \tag{25}$$

one can write the Fourier transform of $|\eta\rangle$ as

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\eta_2}{\sqrt{2\pi}}|\eta\rangle\,\mathrm{e}^{-\mathrm{i}u\eta_2} = \left|\frac{u+\eta_1}{\sqrt{2}}\right\rangle_a\left|\frac{u-\eta_1}{\sqrt{2}}\right\rangle_b. \tag{26}$$

Furthermore, if one considers inverse Fourier transform of the above expression, one will get

$$|\eta\rangle = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-\mathrm{i}\eta_1\eta_2}\int_{-\infty}^{\infty}\mathrm{d}q|q\rangle_a|q - \sqrt{2}\eta_1\rangle_b\,\mathrm{e}^{\mathrm{i}\sqrt{2}\eta_2 q}. \tag{27}$$

This is exactly the well-known Schmidt decomposition of a pure state which expresses that the pure state cannot be factorized as a direct product of two other states and therefore is an entangled state. On the other hand, noting the expression for eigenstate $|p\rangle$ of momentum operator p in the Fock space

$$|p\rangle_a = \frac{1}{\sqrt[4]{\pi}}\exp\left(-\frac{p^2}{2} + \mathrm{i}\sqrt{2}pa^\dagger + \frac{a^{\dagger 2}}{2}\right)|0\rangle, \tag{28}$$

one can also derive

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\eta_1}{\sqrt{2\pi}}|\eta\rangle\,\mathrm{e}^{\mathrm{i}v\eta_1} = \left|\frac{v+\eta_2}{\sqrt{2}}\right\rangle_a\left|\frac{-v+\eta_2}{\sqrt{2}}\right\rangle_b \tag{29}$$

in terms of the eigenstates of the momentum operator whose inverse Fourier transform leads to another standard expression for an entangled state,

$$|\eta\rangle = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-\mathrm{i}\eta_1\eta_2}\int_{-\infty}^{\infty}\mathrm{d}p|p + \sqrt{2}\eta_2\rangle_a|-p\rangle_b\,\mathrm{e}^{-\mathrm{i}\sqrt{2}\eta_1 p}. \tag{30}$$

For the eigenstate $|\xi\rangle$, using the eigenstate $|q\rangle$ of the coordinate operator (equation (25)), one has similarly

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\xi_2}{\sqrt{2\pi}}|\xi\rangle\,\mathrm{e}^{-\mathrm{i}u\xi_2} = \left|\frac{u+\xi_1}{\sqrt{2}}\right\rangle_a\left|\frac{u-\xi_1}{\sqrt{2}}\right\rangle_b. \tag{31}$$

Its inverse Fourier transform is the Schmidt decomposition of the state $|\xi\rangle$

$$|\xi\rangle = \frac{1}{\sqrt{\pi}} \mathrm{e}^{\mathrm{i}\xi_1\xi_2} \int_{-\infty}^{\infty} \mathrm{d}q |q + \sqrt{2}\xi_1\rangle_a |-q\rangle_b \, \mathrm{e}^{\mathrm{i}\sqrt{2}\xi_2 q}. \tag{32}$$

Of course, in terms of the eigenstate $|p\rangle$ of the momentum operator (equation (28)), one can get

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi_1}{\sqrt{2\pi}} |\xi\rangle \, \mathrm{e}^{\mathrm{i}v\xi_1} = \left|\frac{v+\xi_2}{\sqrt{2}}\right\rangle_a \left|\frac{v-\xi_2}{\sqrt{2}}\right\rangle_b, \tag{33}$$

and its inverse transform leads to another Schmidt decomposition of the state $|\xi\rangle$

$$|\xi\rangle = \frac{1}{\sqrt{\pi}} \mathrm{e}^{\mathrm{i}\xi_1\xi_2} \int_{-\infty}^{\infty} \mathrm{d}p |p\rangle_a |p - \sqrt{2}\xi_2\rangle_b \, \mathrm{e}^{-\mathrm{i}\sqrt{2}\xi_1 p}. \tag{34}$$

Therefore it is reasonable to call the $|\eta\rangle$ and $|\xi\rangle$ representations as entangled state representations.

4. Unitary transformations

In the past few years NCQM was discussed extensively from various aspects [8]. The most popular method of formulating NCQM, in the vast literature, is treating the coordinate operators (such as $\hat{X}$ and $\hat{Y}$) as commuting (usually denoting them as x and y respectively), but introducing $*_\theta$-product (for instance, in the non-commutative plane, $*_\theta \equiv \exp\left(\frac{\mathrm{i}\theta}{2}(\overleftarrow{\partial}_x \overrightarrow{\partial}_y - \overleftarrow{\partial}_y \overrightarrow{\partial}_x)\right)$ between functions on the non-commutative space to reflect the non-commutativity of coordinates. For example, using the $*_\theta$-product, Schrödinger equation

$$\hat{H}(\hat{X}, \hat{Y}, \hat{P}_x, \hat{P}_y)|\psi\rangle = E|\psi\rangle \tag{35}$$

on the non-commutative plane should be written as [9]

$$\hat{H}(x, y, p_x, p_y) *_\theta \psi(x, y) = E\psi(x, y) \tag{36}$$

where $\psi(x, y) = \langle x, y|\psi\rangle$, the operators $\hat{X}$, $\hat{Y}$, $\hat{P}_x$ and $\hat{P}_y$ satisfy the commutation relations (1), and the operators x, y, p_x and p_y satisfy the commutation relations (4), respectively. In fact, in equation (36) the representation $|x, y\rangle = |x\rangle_a|y\rangle_b$ has been used, which is the common eigenstate of the operators x and y (not operators $\hat{X}$ and $\hat{Y}$), so we would name it the 'commuting coordinate representation'. In other words, using the $*_\theta$-product in NCQM is equivalent to using the transformations (3), which is also equivalent to using the representation $|x, y\rangle$. In the $|x, y\rangle$ representation one can simply write the matrix elements of the operators $\hat{X}$, $\hat{Y}$, $\hat{P}_x$ and $\hat{P}_y$ on the non-commutative plane:

$$\begin{aligned}
\langle x, y|\hat{X}|x', y'\rangle &= \left(x + \frac{\mathrm{i}\theta}{2}\partial_y\right)\delta(x-x')\delta(y-y'), \\
\langle x, y|\hat{Y}|x', y'\rangle &= \left(y - \frac{\mathrm{i}\theta}{2}\partial_x\right)\delta(x-x')\delta(y-y'), \\
\langle x, y|\hat{P}_x|x', y'\rangle &= -\mathrm{i}\partial_x\delta(x-x')\delta(y-y'), \\
\langle x, y|\hat{P}_y|x', y'\rangle &= -\mathrm{i}\partial_y\delta(x-x')\delta(y-y').
\end{aligned} \tag{37}$$

From equation (3) we know that the states $|x, y\rangle$ are also common eigenstates of the operators $\hat{X} + \frac{\theta}{2}\hat{P}_y$ and $\hat{Y} - \frac{\theta}{2}\hat{P}_x$, so there is a unitary transformation between the two representations ($|\eta\rangle$ and $|x, y\rangle$) whose matrix elements may be written as

$$\langle\eta|x, y\rangle = \frac{1}{\sqrt{\pi}} \mathrm{e}^{\mathrm{i}(\eta_1 - \sqrt{2}x)\eta_2} \delta(x - y - \sqrt{2}\eta_1), \tag{38}$$

where equation (27) is used. Similarly, using equation (32), one can get matrix elements of the transformation between another two representations ($|\xi\rangle$ and $|x, y\rangle$)

$$\langle \xi | x, y\rangle = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-\mathrm{i}(\xi_1 - \sqrt{2}y)\xi_2} \delta(x + y - \sqrt{2}\xi_1). \tag{39}$$

Using the completeness relations (11), (17) and $\int \mathrm{d}x\,\mathrm{d}y |x, y\rangle\langle x, y| = 1$, one can easily see that equations (38) and (39) indeed present the unitary transformations between the entangled state representations and the 'commuting coordinate representation' $|x, y\rangle$.

In order to get a clear form of the unitary transformation between the $|\eta\rangle$ and the $|x, y\rangle$ representations, let us consider the following integration built from the entangled state $|\eta\rangle$ and the two-mode commuting coordinate eigenstate $|x, y\rangle$:

$$\begin{aligned} U &= \int \mathrm{d}^2\eta |x, y\rangle\langle \eta| \big|_{x=\frac{\eta_1+\eta_2}{\sqrt{2}}, y=\frac{\eta_2-\eta_1}{\sqrt{2}}} \\ &= \int \frac{\mathrm{d}\eta_1\,\mathrm{d}\eta_2}{\pi} \exp\left(-\frac{x^2}{2} - \frac{y^2}{2} + \sqrt{2}xa^\dagger + \sqrt{2}yb^\dagger - \frac{a^{\dagger 2}}{2} - \frac{b^{\dagger 2}}{2}\right) \\ &\quad \times |00\rangle\langle 00| \exp\left(-\frac{|\eta|^2}{2} + \eta^* a - \eta b + ab\right)\bigg|_{x=\frac{\eta_1+\eta_2}{\sqrt{2}}, y=\frac{\eta_2-\eta_1}{\sqrt{2}}}. \end{aligned} \tag{40}$$

Here, for simplicity, we have taken all parameters in the ordinary harmonic oscillator expressions (i.e., m, ω) equal to 1. Using the trick in the calculation of equation (11), we obtain

$$U =: \exp\left(-\frac{1+\mathrm{i}}{2}(a^\dagger a + b^\dagger b + a^\dagger b + b^\dagger a)\right) :, \tag{41}$$

where the result of the integration is expressed in terms of the normal ordered product. To show the unitary property of the operator U more clearly, introducing an operator $\mathrm{e}^{\mathrm{i}rS}$ with $S = a^\dagger a + b^\dagger b + a^\dagger b + b^\dagger a$, we have $[S, a^\dagger] = a^\dagger + b^\dagger$ and $[S, b^\dagger] = a^\dagger + b^\dagger$, which lead to

$$\mathrm{e}^{\mathrm{i}rS} a^\dagger \mathrm{e}^{-\mathrm{i}rS} = \frac{\mathrm{e}^{2\mathrm{i}r} + 1}{2} a^\dagger + \frac{\mathrm{e}^{2\mathrm{i}r} - 1}{2} b^\dagger, \qquad \mathrm{e}^{\mathrm{i}rS} b^\dagger \mathrm{e}^{-\mathrm{i}rS} = \frac{\mathrm{e}^{2\mathrm{i}r} - 1}{2} a^\dagger + \frac{\mathrm{e}^{2\mathrm{i}r} + 1}{2} b^\dagger, \tag{42}$$

and further

$$\begin{aligned} \mathrm{e}^{\mathrm{i}rS} &= \mathrm{e}^{\mathrm{i}rS} \sum_{n,m=0}^{\infty} |n, m\rangle\langle n, m| \\ &= \mathrm{e}^{\mathrm{i}rS} \sum_{n,m=0}^{\infty} \frac{a^{\dagger n} b^{\dagger m}}{\sqrt{n!m!}} |00\rangle\langle 00| \frac{a^n b^m}{\sqrt{n!m!}} =: \exp\left(-\frac{1 - \mathrm{e}^{2\mathrm{i}r}}{2} S\right) :. \end{aligned} \tag{43}$$

When choosing $r = -\pi/4$, we have

$$U = \exp\left(-\frac{\mathrm{i}\pi}{4}(a^\dagger a + b^\dagger b + a^\dagger b + b^\dagger a)\right), \tag{44}$$

which is unitary clearly. From equation (44), it is easy to obtain

$$U a^\dagger U^\dagger = \frac{1 - \mathrm{i}}{2} a^\dagger - \frac{1 + \mathrm{i}}{2} b^\dagger, \qquad U b^\dagger U^\dagger = -\frac{1 + \mathrm{i}}{2} a^\dagger + \frac{1 - \mathrm{i}}{2} b^\dagger, \tag{45}$$

which lead to

$$U|\eta\rangle = |x, y\rangle|_{x=\frac{\eta_1+\eta_2}{\sqrt{2}}, y=\frac{\eta_2-\eta_1}{\sqrt{2}}}, \qquad U|x, y\rangle = |\eta^*\rangle|_{\eta_1=\frac{x-y}{\sqrt{2}}, \eta_2=\frac{x+y}{\sqrt{2}}}. \tag{46}$$

Thus U indeed transform the state $|\eta\rangle$ to the state $|x, y\rangle$ and vice versa. In the $|\eta\rangle$ representation one can calculate the following matrix element of the operator U, $\langle \eta|U|\zeta\rangle$, where $\zeta = \zeta_1 + \mathrm{i}\zeta_2$. Using equation (46) one has

$$\langle \eta|U|\zeta\rangle = \langle \eta|x, y\rangle|_{x=\frac{\zeta_1+\zeta_2}{\sqrt{2}}, y=\frac{\zeta_2-\zeta_1}{\sqrt{2}}} = \frac{1}{\sqrt{\pi}} \mathrm{e}^{\mathrm{i}(\eta_1 - \zeta_1 - \zeta_2)\eta_2} \delta(\sqrt{2}\zeta_1 - \sqrt{2}\eta_1). \tag{47}$$

If one further takes $\zeta_1 = (x-y)/\sqrt{2}$ and $\zeta_2 = (x+y)/\sqrt{2}$, equation (47) will lead to equation (38) exactly. This means that equation (38) is just a matrix element of the unitary operator U in the entangled state representation. Similarly, one can find out the unitary transformation between the $|\xi\rangle$ and the $|x, y\rangle$ representations and endow equation (39) with the same explanation.

Having the unitary transformation (38), and using

$$\langle\eta|\hat{F}|\eta'\rangle = \int \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y'\langle\eta|x, y\rangle\langle x, y|\hat{F}|x', y'\rangle\langle x', y'|\eta'\rangle, \tag{48}$$

or

$$\langle x, y|\hat{F}|x', y'\rangle = \int \mathrm{d}^2\eta\,\mathrm{d}^2\eta'\langle x, y|\eta\rangle\langle\eta|\hat{F}|\eta'\rangle\langle\eta'|x', y'\rangle, \tag{49}$$

one may get the matrix elements of any operator $\hat{F}$ in one representation, if one knows $\hat{F}$ in another representation. For example, taking $\hat{F} = \hat{P}_x$, one has

$$\begin{aligned}\langle\eta|\hat{P}_x|\eta'\rangle &= \int \frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}x'\,\mathrm{d}y'}{\pi}\,\mathrm{e}^{\mathrm{i}(\eta_1-\sqrt{2}x)\eta_2}\delta(x-y-\sqrt{2}\eta_1)(-\mathrm{i}\partial_x)\delta(x-x')\delta(y-y')\\ &\quad\times \mathrm{e}^{-\mathrm{i}(\eta_1'-\sqrt{2}x')\eta_2'}\delta(x'-y'-\sqrt{2}\eta_1')\\ &= \mathrm{e}^{\mathrm{i}\eta_1(\eta_2-\eta_2')}\left(\sqrt{2}\eta_2 - \mathrm{i}\frac{\partial}{\partial\sqrt{2}\eta_1}\right)\delta(\eta_1-\eta_1')\delta(\eta_2-\eta_2')\\ &= \frac{1}{\sqrt{2}}\left(\eta_2 - \mathrm{i}\partial_{\eta_1}\right)\delta^2(\eta-\eta')\end{aligned} \tag{50}$$

which exactly coincides with equation (23). Similarly, one also has

$$\begin{aligned}\langle\eta|\hat{P}_y|\eta'\rangle &= \mathrm{e}^{\mathrm{i}\eta_1(\eta_2-\eta_2')}\left(\mathrm{i}\frac{\partial}{\partial\sqrt{2}\eta_1}\right)\delta(\eta_1-\eta_1')\delta(\eta_2-\eta_2')\\ &= \frac{1}{\sqrt{2}}(\eta_2 + \mathrm{i}\partial_{\eta_1})\delta^2(\eta-\eta').\end{aligned} \tag{51}$$

Noting that

$$x\,\mathrm{e}^{\mathrm{i}(\eta_1-\sqrt{2}x)\eta_2} = \frac{1}{\sqrt{2}}(\eta_1 + \mathrm{i}\partial_{\eta_2})\,\mathrm{e}^{\mathrm{i}(\eta_1-\sqrt{2}x)\eta_2}, \tag{52}$$

one can obtain other two expressions of equation (23). Of course, with the aid of the unitary transformation (39), from equation (37) one may get equation (24).

Therefore, we derive the unitary transformations which change the $|x, y\rangle$ representation to the $|\eta\rangle$ (or similarly to the $|\xi\rangle$) representation and vice versa.

5. An example of a coupled harmonic oscillator

It is well known that representation plays a basic role in quantum mechanics like the coordinate systems in geometry. In section 2 we introduced the entangled state representations $|\eta\rangle$ and $|\xi\rangle$, which are related to the $|x, y\rangle$ representation by unitary transformations as shown in section 4. In the $|\eta\rangle$ or $|\xi\rangle$ representation one can also solve the Schrödinger equation of NCQM as in the $|x, y\rangle$ representation, and sometimes it is more convenient working in the entangled state representation than in the $|x, y\rangle$ representation. To show this, let us study a two-dimensional harmonic oscillator on the non-commutative plane with both momentum–momentum (kinetic) coupling and coordinate–coordinate (elastic) coupling. The Hamiltonian is

$$H = \frac{1}{2}\hat{P}_x^2 + \frac{1}{2}\hat{P}_y^2 + \frac{1}{2}\hat{X}^2 + \frac{1}{2}\hat{Y}^2 + \kappa\hat{P}_x\hat{P}_y + \frac{\lambda}{2}(\hat{X}\hat{Y} + \hat{Y}\hat{X}), \tag{53}$$

where the operators $\hat{P}_x$, $\hat{P}_y$, $\hat{X}$ and $\hat{Y}$ satisfy the commutation relations (1). After substituting equation (3) into equation (53) we get the Hamiltonian H in the $|x, y\rangle$ representation,

$$H = \frac{1}{2}\left(1+\frac{\theta^2}{4}\right)p_x^2 + \frac{1}{2}\left(1+\frac{\theta^2}{4}\right)p_y^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + \left(\kappa - \frac{\lambda\theta^2}{4}\right)p_x p_y + \lambda xy - \frac{\theta}{2}(xp_y - yp_x) + \frac{\lambda\theta}{2}(xp_x - yp_y), \tag{54}$$

which includes not only the kinetic and the elastic coupling terms, but also the coordinate–momentum coupling terms (they are the angular momentum term and the squeezing term, respectively). It is not an easy task to solve its eigenequation. However, in the $|\eta\rangle$ representation the Hamiltonian H has a simpler form,

$$H = \frac{1}{2}\left(1+\frac{\theta^2}{2} - \kappa + \frac{\lambda\theta^2}{4}\right)p_1^2 + \frac{1}{2}(1+\lambda)p_2^2 - \frac{\theta}{2}(1+\lambda)p_1p_2 + \frac{1}{2}(1-\lambda)\eta_1^2 + \frac{1}{2}\left(1+\frac{\theta^2}{2}+\kappa-\frac{\lambda\theta^2}{4}\right)\eta_2^2 - \frac{\theta}{2}(1-\lambda)\eta_1\eta_2, \tag{55}$$

where $p_i = -\mathrm{i}\partial/\partial\eta_i$ $(i = 1, 2)$. In the Hamiltonian (55), only the kinetic and the elastic coupling terms survive, and it is easier to handle than the form (54). Of course, it is needless to emphasize that Hamiltonians (54) and (55) are connected via a unitary transformation described in section 4.

Before diagonalizing H (55), let us introduce the notations to rewrite (55) so that it has a more familiar form:

$$m_1 = \left(1+\frac{\theta^2}{2}-\kappa+\frac{\lambda\theta^2}{4}\right)^{-1}, \quad \omega_1 = \sqrt{(1-\lambda)\left(1+\frac{\theta^2}{2}-\kappa+\frac{\lambda\theta^2}{4}\right)}, \quad \alpha = \frac{\theta}{2}(1+\lambda),$$
$$m_2 = (1+\lambda)^{-1}, \quad \omega_2 = \sqrt{(1+\lambda)\left(1+\frac{\theta^2}{2}+\kappa-\frac{\lambda\theta^2}{4}\right)}, \quad \beta = \frac{\theta}{2}(1-\lambda). \tag{56}$$

In terms of these notations, H (55) becomes

$$H = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \alpha p_1 p_2 + \frac{m_1\omega_1^2}{2}\eta_1^2 + \frac{m_2\omega_2^2}{2}\eta_2^2 - \beta\eta_1\eta_2. \tag{57}$$

Now let us introduce a 2×2 matrix A whose matrix elements a_{ij} will be determined later $(i, j = 1, 2)$. If we use $\vec{p}$ to denote the two-dimensional momentum (p_1, p_2), one can write $\vec{\tilde{p}} = A\vec{p} = (\tilde{p}_1, \tilde{p}_2)$ with $\tilde{p}_i = a_{ij}p_j$, and inversely, $p_i = b_{ij}\tilde{p}_j$, where b_{ij} are the elements of the inverse matrix of A. Consider the following transformation

$$V = \sqrt{\det A}\int \mathrm{d}\vec{p}\,|A\vec{p}\rangle\langle\vec{p}| \tag{58}$$

in the Hilbert space spanned by two-mode momentum eigenstates $|\vec{p}\rangle$, which is unitary clearly,

$$VV^\dagger = \det A\int \mathrm{d}\vec{p}\,\mathrm{d}\vec{p}\,'|A\vec{p}\rangle\langle\vec{p}|\vec{p}\,'\rangle\langle A\vec{p}\,'| = \det A\int \mathrm{d}\vec{p}\,|A\vec{p}\rangle\langle A\vec{p}| = \int \mathrm{d}\vec{\tilde{p}}\,|\vec{\tilde{p}}\rangle\langle\vec{\tilde{p}}| = 1, \tag{59}$$

and similarly $V^\dagger V = 1$. In equation (58), $|\vec{p}\rangle = |p_1\rangle|p_2\rangle$ and $|p_i\rangle$ are the momentum eigenstates

$$|p_i\rangle = \left(\frac{1}{\pi m_i\omega_i}\right)^{1/4}\exp\left(-\frac{p_i^2}{2m_i\omega_i} + \mathrm{i}\sqrt{\frac{2}{m_i\omega_i}}p_i a_i^\dagger + \frac{1}{2}a_i^{\dagger 2}\right)|0\rangle_i, \tag{60}$$

where $a_i^\dagger$ (and a_i) are the ordinary bosonic creation (and annihilation) operators

$$a_i = \frac{1}{2}\left(\sqrt{m_i\omega_i}\eta_i + \mathrm{i}\frac{1}{\sqrt{m_i\omega_i}}p_i\right), \qquad a_i^\dagger = \frac{1}{2}\left(\sqrt{m_i\omega_i}\eta_i - \mathrm{i}\frac{1}{\sqrt{m_i\omega_i}}p_i\right). \tag{61}$$

It is not difficult to see that V transforms $Vp_iV^\dagger = b_{ij}p_j$ and $V\eta_iV^\dagger = a_{ji}\eta_j$, because

$$Vp_iV^\dagger = \det A\int \mathrm{d}\vec{p}\,\mathrm{d}\vec{p}'|A\vec{p}\rangle\langle\vec{p}|p_i|\vec{p}'\rangle\langle A\vec{p}'| = \int \mathrm{d}\vec{\tilde{p}}b_{ij}\tilde{p}_j|\vec{\tilde{p}}\rangle\langle\vec{\tilde{p}}| = b_{ij}p_j, \tag{62}$$

and

$$V\eta_iV^\dagger = \det A\int \mathrm{d}\vec{p}\,\mathrm{d}\vec{p}'|A\vec{p}\rangle\mathrm{i}\frac{\partial}{\partial p_i}\delta(\vec{p}-\vec{p}')\langle A\vec{p}'|. \tag{63}$$

Furthermore, acting equation (63) from the right-hand side on $\langle\vec{\eta}|$ leads to

$$\langle\vec{\eta}|V\eta_iV^\dagger = \det A\int \mathrm{d}\vec{p}\,\mathrm{d}\vec{p}'\left(-\mathrm{i}\frac{\partial}{\partial p_i}\exp(\mathrm{i}a_{jk}p_k\eta_j)\right)\delta(\vec{p}-\vec{p}')\langle A\vec{p}'| = \langle\vec{\eta}|a_{ji}\eta_j, \tag{64}$$

which means that $V\eta_iV^\dagger = a_{ji}\eta_j$.

Now let us act the unitary transformation V on the Hamiltonian (57) and get

$$\begin{aligned} VHV^\dagger = {}& \frac{1}{2m_1}(b_{11}p_1 + b_{12}p_2)^2 + \frac{1}{2m_2}(b_{21}p_1 + b_{22}p_2)^2 - \alpha(b_{11}p_1 + b_{12}p_2)(b_{21}p_1 + b_{22}p_2) \\ & + \frac{m_1\omega_1^2}{2}(a_{11}\eta_1 + a_{21}\eta_2)^2 + \frac{m_2\omega_2^2}{2}(a_{12}\eta_1 + a_{22}\eta_2)^2 \\ & - \beta(a_{11}\eta_1 + a_{21}\eta_2)(a_{12}\eta_1 + a_{22}\eta_2). \end{aligned} \tag{65}$$

Then in order to annihilate the coupling terms in equation (65), we set

$$\begin{aligned} & \frac{1}{m_1}a_{22}a_{12} + \frac{1}{m_2}a_{21}a_{22} + \alpha(a_{11}a_{22} + a_{12}a_{21}) = 0, \\ & m_1\omega_1^2a_{11}a_{21} + m_2\omega_2^2a_{12}a_{22} - \beta(a_{11}a_{22} + a_{12}a_{21}) = 0. \end{aligned} \tag{66}$$

From equation (66) we have

$$a_{12} = \frac{\Omega m_1}{2\alpha\left(\beta + \alpha m_1m_2\omega_2^2\right)}a_{11}, \qquad a_{21} = -\frac{\Omega m_2}{2\alpha\left(\beta + \alpha m_1m_2\omega_1^2\right)}a_{22}, \tag{67}$$

where

$$\Omega = \alpha\left(\omega_1^2 - \omega_2^2\right) + \sqrt{\alpha^2\left(\omega_1^2 - \omega_2^2\right)^2 + 4\alpha^2\left(\frac{\beta}{m_1} + \alpha m_2\omega_2^2\right)\left(\frac{\beta}{m_2} + \alpha m_1\omega_1^2\right)}. \tag{68}$$

Thus equation (65) can be written as

$$\begin{aligned} H_d = {}& \frac{a_{22}^2}{2m_1(\det A)^2}\left(1 + \frac{m_1m_1\Omega}{\beta + \alpha m_1m_2\omega_1^2} + \frac{m_1m_2\Omega^2}{4\alpha^2\left(\beta + \alpha m_1m_2\omega_1^2\right)^2}\right)p_1^2 \\ & + \frac{a_{11}^2}{2m_2(\det A)^2}\left(1 - \frac{m_1m_1\Omega}{\beta + \alpha m_1m_2\omega_2^2} + \frac{m_1m_2\Omega^2}{4\alpha^2\left(\beta + \alpha m_1m_2\omega_2^2\right)^2}\right)p_2^2 \\ & + \frac{a_{11}^2m_1\omega_1^2}{2}\left(1 - \frac{\beta\Omega}{\alpha\omega_1^2\left(\beta + \alpha m_1m_2\omega_2^2\right)} + \frac{m_1m_2\omega_2^2\Omega^2}{4\alpha^2\omega_1^2\left(\beta + \alpha m_1m_2\omega_2^2\right)^2}\right)\eta_1^2 \\ & + \frac{a_{22}^2m_2\omega_2^2}{2}\left(1 + \frac{\beta\Omega}{\alpha\omega_2^2\left(\beta + \alpha m_1m_2\omega_1^2\right)} + \frac{m_1m_2\omega_1^2\Omega^2}{4\alpha^2\omega_2^2\left(\beta + \alpha m_1m_2\omega_1^2\right)^2}\right)\eta_2^2. \end{aligned} \tag{69}$$

Since

$$\det A = a_{11}a_{22} - a_{12}a_{21} = a_{11}a_{22}\left(1 + \frac{m_1 m_2 \Omega^2}{4\alpha^2\left(\beta + \alpha m_1 m_2 \omega_1^2\right)\left(\beta + \alpha m_1 m_2 \omega_2^2\right)}\right), \tag{70}$$

if we use the following notations,

$$\begin{aligned}
T_1 &= 1 + \frac{m_1 m_2 \Omega^2}{4\alpha^2\left(\beta + \alpha m_1 m_2 \omega_1^2\right)\left(\beta + \alpha m_1 m_2 \omega_2^2\right)}, \\
T_2 &= 1 + \frac{m_1 m_1 \Omega}{\beta + \alpha m_1 m_2 \omega_1^2} + \frac{m_1 m_2 \Omega^2}{4\alpha^2\left(\beta + \alpha m_1 m_2 \omega_1^2\right)^2}, \\
T_3 &= 1 - \frac{\beta\Omega}{\alpha\omega_1^2\left(\beta + \alpha m_1 m_2 \omega_2^2\right)} + \frac{m_1 m_2 \omega_2^2 \Omega^2}{4\alpha^2\omega_1^2\left(\beta + \alpha m_1 m_2 \omega_2^2\right)^2}, \\
T_4 &= 1 - \frac{m_1 m_1 \Omega}{\beta + \alpha m_1 m_2 \omega_2^2} + \frac{m_1 m_2 \Omega^2}{4\alpha^2\left(\beta + \alpha m_1 m_2 \omega_2^2\right)^2}, \\
T_5 &= 1 + \frac{\beta\Omega}{\alpha\omega_2^2\left(\beta + \alpha m_1 m_2 \omega_1^2\right)} + \frac{m_1 m_2 \omega_1^2 \Omega^2}{4\alpha^2\omega_2^2\left(\beta + \alpha m_1 m_2 \omega_1^2\right)^2},
\end{aligned} \tag{71}$$

and in equation (69), let the coefficients of the terms $p_1^2/2m_1$ and $p_2^2/2m_2$ be equal to the coefficients of the terms $m_1\omega_1^2\eta_1^2/2$ and $m_2\omega_2^2\eta_2^2/2$ respectively, and denote them as Λ_1 and Λ_2, we have

$$\begin{aligned}
a_{11} &= T_1^{-1/2} T_2^{1/4} T_3^{-1/4}, \qquad & a_{22} &= T_1^{-1/2} T_4^{1/4} T_5^{-1/4}, \\
\Lambda_1 &= T_1^{-1} T_2^{1/2} T_3^{1/2}, \qquad & \Lambda_2 &= T_1^{-1} T_4^{1/2} T_5^{1/2}.
\end{aligned} \tag{72}$$

Thus we diagonalize the Hamiltonian (55) and obtain

$$H_d = VHV^\dagger = \Lambda_1\omega_1\left(a_1^\dagger a_1 + \tfrac{1}{2}\right) + \Lambda_2\omega_2\left(a_2^\dagger a_2 + \tfrac{1}{2}\right), \tag{73}$$

which gives the energy spectrum of the two-dimensional harmonic oscillator (53) on the non-commutative plane with both the kinetic and the elastic couplings

$$E_{n,m} = \Lambda_1\omega_1\left(n + \tfrac{1}{2}\right) + \Lambda_2\omega_2\left(m + \tfrac{1}{2}\right). \tag{74}$$

This result, to our knowledge, has not been reported in the literature so far. In some special case, however, it reduces to well-known relevant results. For example, when the coupling constants κ and λ both vanish, the Hamiltonian (53) describes a two-dimensional harmonic oscillator without any coupling on the non-commutative plane. Equation (74) reduces to

$$E_{n,m} = \sqrt{1 + \frac{\theta^2}{4}}(n + m + 1), \tag{75}$$

which was derived by many authors in other methods. For instance, equation (75) coincides with [10].

6. Summary and discussion

In order to develop representation and transformation theory so that one can solve more dynamic problems for NCQM, in this work we introduce new representations on the non-commutative space which may be named the entangled state representations, because the state vectors of these representations are common eigenstates of the difference (or the sum) of two different coordinate-component operators and the sum (or the difference) of two relevant momentum operators, and display some entanglements of different components on the non-commutative space. Since these state vectors are orthonormal and satisfy the completeness relation, they form representations to formulate the NCQM. In this work, we find out an explicit unitary operator which can transform the entangled state representation $|\eta\rangle$ into the 'commuting coordinate representation' $|x, y\rangle$ used in the literature on NCQM. A similar unitary operator between the $|\xi\rangle$ representation and the $|x, y\rangle$ representation can also be obtained. To show the potential applications of new entangled representations, we solve exactly a two-dimensional harmonic oscillator with both the kinetic and the elastic couplings on the non-commutative plane. This example shows that some dynamic problems of NCQM may be easily solved in the entangled state representations.

It is also interesting to generalize the entangled state representations to describe two particles moving on the non-commutative space. Work in this direction will be presented in a separate paper.

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